# Formation control of force-controlled mobile robots in a spanning-tree topology

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***Abstract*— We solve the formation-tracking control problem for mobile robots via linear control, under the assumption that each agent communicates only with one "leader" robot and with one follower. We assume that the system is force-controlled (hence we use the *dynamic* model) as opposed to velocity-controlled (in the kinematic-model case). As in the classical tracking control problem for nonholonomic systems, the swarm is driven by a fictitious robot which moves about freely and which is leader to one robot only. For the case of a fixed spanning-tree topology we show that persistency of excitation on the velocity of the virtual leader is sufficient and necessary to achieve consensus tracking.**

## I. Introduction

In the last decade, coordinated control of autonomous mobile robots has received great attention motivated by the fact that a group of robots may accomplish certain tasks with greater efficiency, flexibility, robustness and safety than a single robot. However, coordinated motion requires more complex control schemes as well as path planning; for instance, it may be achieved through local individual tracking control on each robot provided that all agents communicate with each other. Furthermore, in many applications such as search & rescue, surveillance or transportation, a group of mobile robots is supposed to follow a predefined trajectory while maintaining a desired formation shape.

There are various formation-control methods proposed in the literature such as the behavior approach [1], [2], the virtual structure method [3], [4], the graph-theory approach as in [5], [6], [7], *etc.*

The leader-follower approach as in [8], [9] is reminiscent of the so-called master-slave synchronization paradigm. Extended to the case of more than two agents, one or more vehicles may be considered as *leader* and the rest of the robots are considered *followers* as they are required to track their leaders' trajectories with a predefined formation shape. In the context of mobile robots, a virtual reference vehicle is assumed as a leader over all the rest. From a graph viewpoint, it is the reference vehicle which plays the role of a *root* node. Leaders are children of the root node that is, robots which "know" the reference trajectory. All other nodes are either followers and leaders simultaneously (intermediate nodes in the graph topology) or followers (leaves, nodes without children in the graph topology). The method is easy to understand and to implement. It is scalable for any number of agents. There is no explicit feedback from followers to leaders (the graph is directed) but followers require full state information of their leaders.

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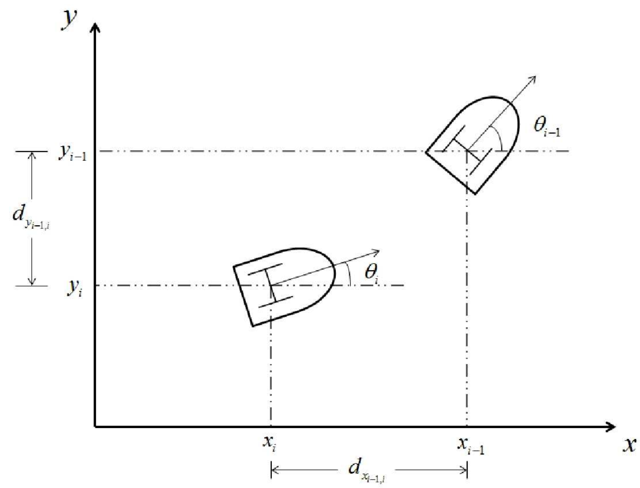


Fig. 1. Generic representation of a leader-follower configuration. For a swarm of $n$ vehicles, any geometric topology may be easily defined by determining the position of each vehicle relative to its leader. This does not affect the kinematic model.

In [10], an adaptive leader-follower based formation control without the need of leaders' velocity information is proposed. It is assumed that two robots act as leaders hence, they know the prescribed reference velocity, while the others are considered to be followers, with single integrator dynamics. A stability analysis shows that the triangular formation is asymptotically stable while the co-linear one is not. In [11], the authors present a three-level hybrid control architecture based on feedback linearization; the analysis relies on graph theory. It shows that position error system is asymptotically stable with a bounded orientation error. The method is supported by both simulations and experimental results. In [12], a virtual vehicle is designed to eliminate velocity measurement of the leader then using backstepping and Lyapunov's direct method position tracking control problem of the follower is solved. The proposed method guarantees asymptotic stability of the closed loop error system dynamic. Another asymptotic stability result is presented in [13]. Proposed control strategy ensures follower position to vary in proper circle arcs centered in the leader's reference frame satisfying suitable input constraints. In [14], the leader's influence on the trajectory tracking error dynamics is taken as an unknown but bounded, observable disturbance and eliminated by the local controllers of followers; ultimate boundedness of the trajectory errors is established. In [15], consensus protocols under directed communication topology are designed using time-varying consensus gains to reduce the noise effects. Strong mean square consensus is provided through algebraic graph theory and stochastic analysis.

In this paper, we follow a leader-follower approach; we assume that the swarm of $n$ vehicles has only one leader which communicates with the virtual reference vehicle that is, only one robot "knows" the reference trajectory. The formation is ensured via a one-to-one unilateral communication that is, each robot except for the leader (root agent) and the last follower (tail agent), communicates only with one follower and with one leader. To the former the robot gives information of its full state, from the latter it receives full state information which is taken by the decentralized controller as a reference. The communication graph is directed that is, there exist no feedback from followers to leaders. Tail agents are robots with no followers (the leaves in the graph tree).

Our controllers are inspired by similar controllers previously reported for tracking control of a single robot –see [16]. The control design and therefore, the stability analysis problems are divided into the tracking control for the translation variables and tracking of the heading angle. This separation-principle approach leads to fairly simple controllers, linear time-varying. The analysis relies on the ability to study the behavior of the translational errors and heading errors separately. For the former, it is established that a sufficient and necessary condition is that the reference angular trajectory of the virtual leader robot have the property of persistency of excitation, for the heading angles, a simple proportional feedback is enough. The analysis of the over-all closed loop system relies on tracking theorems tailored for so-called cascaded (time-varying) systems. The significance of the proof method relies in the circumvention of graph theory, eigen-value analysis and other tools difficult to extend to the realm of nonlinear systems.

In Section II we formulate the control problem; our main result is presented in Section III; numerical simulation results are provided in Section IV and we close with some concluding remarks in Section V.

## II. Problem formulation and its solution

Consider a group of $n$ unicycle robots modelled by

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) \\ \dot{y}_i &= v_i \sin(\theta_i) \\ \dot{\theta}_i &= w_i \\ \dot{v}_i &= \frac{u_{1i}}{m_i} \\ \dot{w}_i &= \frac{u_{2i}}{j_i} \end{aligned} \tag{1}$$

with $i \in [1, n]$. In contrast to the kinematic model –see [17] which is velocity-controlled the control inputs $u_{1i}$ and $u_{2i}$ correspond to force and torque respectively; $m_i$ denotes the mass of the $i$th robot, while and $j_i$ stands for the moment of inertia. The coordinates $x_i$ and $y_i$ represent the center of the $i^{th}$ mobile robot with respect to a globally-fixed frame and $\theta_i$ is the heading angle –see Figure 1. The linear and angular velocities of the $i^{th}$ robot are denoted $v_i$ and $w_i$ respectively. The control objective is to make the $n$ robots take specific postures determined by the topology designer, and to make the swarm follow a path determined by the virtual reference vehicle $R_0$. Mostly any geometrical configuration may be achieved and one can choose any point in the Cartesian plane to follow the virtual reference vehicle. The swarm has only one 'leader' robot tagged $R_1$ whose local controller uses knowledge of the reference trajectory generated by the virtual leader; in the communications graph, $R_1$ is the child of the root-node robot $R_0$. The other robots are intermediate robots labeled $R_2$ to $R_{n-1}$ that is, $R_i$ acts as leader for $R_{i+1}$ and follows $R_{i-1}$. The last robot in the communication topology is denoted $R_n$ and has no followers that is, it constitutes the tail node of the spanning tree –see Figure 2. We remark that the notation $R_{i-1}$ refers to the *graph* topology as illustrated in Figure 2 but it does not determine a physical formation.

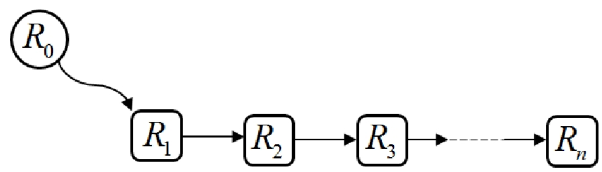


Fig. 2. Communication topology: a spanning directed tree with permanent communication between $R_i$ and $R_{i+1}$ for all $i \in [0, n-1]$.

To reformulate the control goal as a stabilization problem, we follow [17] and define the tracking errors

$$\begin{aligned} p_{ix} &= x_{i-1} - x_i - d_{x(i-1),i} \\ p_{iy} &= y_{i-1} - y_i - d_{y(i-1),i} \\ p_{i\theta} &= \theta_{i-1} - \theta_i \end{aligned}$$

where $d_{x(i-1),i}$ and $d_{y(i-1),i}$ denote the desired distances between any pair leader-follower robots. Note that for the leader robot $(R_1)$ and the reference virtual robot $(R_0)$ these values are set to zero $(d_{x0,1} = d_{y0,1} = 0)$. In addition, for simplicity we assume here that all robots are to be aligned with the same heading $(d_{i\theta} = 0)$ for all $i \in \{2, ...n\}$. Then we translate the tracking errors from the global coordinate frame to local coordinates fixed on the robot that is, let

$$\begin{bmatrix} e_{ix} \\ e_{iy} \\ e_{i\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{i\theta} \end{bmatrix}. \tag{2}$$

Furthermore, we define the velocity error variables

$$\begin{aligned} e_{iv} &= v_i - v_{i-1} \\ e_{iw} &= w_i - w_{i-1}\,. \end{aligned} \tag{3}$$

Then, the dynamics of the error trajectories between any pair of robots $R_{i-1}$ followed by $R_i$ is given by

$$\begin{aligned}
\dot{e}_{ix} &= w_{(i-1)}e_{iy} - v_{i-1} + v_{i-1}\cos e_{i\theta} - e_{iv} + e_{iw}e_{iy} && \text{(4a)}\\
\dot{e}_{iy} &= -w_{(i-1)}e_{ix} + v_{(i-1)}\sin e_{i\theta} - e_{iw}e_{ix} && \text{(4b)}\\
\dot{e}_{i\theta} &= w_{i-1} - (e_{iw} + w_{i-1}) && \text{(4c)}\\
\dot{e}_{iv} &= \frac{u_{1i}}{m_i} - \dot{v}_{i-1} && \text{(4d)}\\
\dot{e}_{iw} &= \frac{u_{2i}}{j_i} - \dot{w}_{i-1}. && \text{(4e)}
\end{aligned}$$

The consensus formation-tracking control problem comes to stabilize all the error systems (4) at the origin that is, for all $i \in [1, n]$. More precisely, the objective is to find a control law $u_i = [u_{1i}, u_{2i}]^T$ of the form

$$\begin{aligned}
u_{1i} &= u_{1i}\left(t, e_{ix}, e_{iy}, e_{i\theta}, v, w\right)\\
u_{2i} &= u_{2i}\left(t, e_{ix}, e_{iy}, e_{i\theta}, v, w\right) && \text{(5)}
\end{aligned}$$

such that the closed loop error dynamics is uniformly globally exponentially stable. The approach that we present is based on cascades-based control, it consists in decoupling via feedback, the translational error dynamics from the heading error dynamics.

Generally speaking, cascaded-based control relies on the ability to design controllers so that the closed-loop system has a cascaded structure,

$$\begin{aligned}
\dot{x}_1 &= f_1\left(t, x_1\right) + g\left(t, x\right) && \text{(6a)}\\
\dot{x}_2 &= f_2\left(t, x_2\right) && \text{(6b)}
\end{aligned}$$

–note that the lower dynamics (6b) is independent of the variable $x_1$ and the dynamic equation corresponding to the latter is "perturbed" by $x_2$ through the *interconnection* term $g\left(t, x\right)$, hence the term *cascade*. Stability of the origin of the cascaded system may be asserted by relying on [18], [Lemma 3] which establishes that the origin of a cascaded system is uniformly globally asymptotically stable if so are the respective origins of the disconnected subsystems that is, when the interconnection $g \equiv 0$ and if the solutions of the perturbed dynamics (6a) remain bounded. In the appendix we present a concrete stability theorem whose conditions serve as guidelines for control design and fits the purposes of this paper.

In that regard, it is important to stress that the error dynamics (4) already possesses a cascaded structure, with $x_1 = [e_x,\, e_v,\, e_y]^\top$ and $x_2 = [e_\theta, e_w]^\top$; indeed, the latter may be regarded as an input generating a perturbation to the translational dynamics equations (4a), (4b) and (4d). With this in mind, we follow the approach originally proposed in [16], where uniform global exponential stability was first established for the tracking control problem[1]. Our main result establishes that the rationale used in tracking control, may be applied to solve the problem of formation control. We show that a linear time-varying controller applied locally on each robot suffices to solve the formation-tracking control paradigm.

[1]See [19] for several extensions inspired by the main results in [16].

## III. MAIN RESULTS

Let each local controller be defined by

$$\begin{aligned}
u_{1i} &= m_i\left(\dot{v}_{i-1} + c_{3i}e_{ix} - c_{4i}e_{iv}\right) && \text{(7a)}\\
u_{2i} &= j_i\left(\dot{w}_{i-1} + c_{5i}e_{i\theta} - c_{6i}e_{iw}\right) && \text{(7b)}
\end{aligned}$$

–note that this controller requires the knowledge of $u_{1(i-1)}$ and $u_{2(i-1)}$; these do not need to be computed by the $i$th robot but their value may be received as a measurement, from the leading robot $R_{i-1}$. Define $e_w := [e_{1w} \cdots e_{nw}]^\top$, and similarly for $e_x$, $e_y$, $e_\theta$, $e_v$. Then, replacing (7) in (4) and using $w_{i-1} = w_i - e_{iw}$ we obtain by direct computation,

$$\begin{aligned}
\dot{e}_{ix} &= w_i(t, e_w)e_{iy} - e_{iv} + v_{i-1}[\cos e_{i\theta} - 1] && \text{(8a)}\\
\dot{e}_{iy} &= -w_i(t, e_w)e_{ix} + v_{(i-1)}\sin e_{i\theta} && \text{(8b)}\\
\dot{e}_{i\theta} &= -e_{iw} && \text{(8c)}\\
\dot{e}_{iv} &= c_{3i}e_{ix} - c_{4i}e_{iv} && \text{(8d)}\\
\dot{e}_{iw} &= c_{5i}e_{i\theta} - c_{6i}e_{iw}. && \text{(8e)}
\end{aligned}$$

We stress that $w_i$ is a function of $e_w$ and time, indeed in view of (3) we have $w_1 = e_{1w} + w_0(t)$, $w_2 = e_{2w} + e_{1w} + w_0(t)$ and

$$w_i = e_{iw} + e_{(i-1)w} + \cdots + e_{1w} + w_0(t), \quad \forall\, i \geq 3.$$

The system (8) has a cascades structure in which the translation error dynamics is decoupled from the heading error dynamics. To see this, we first remark that the translation error dynamics may be rewritten in the compact form

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_v \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} 0 & -I & W(t, e_w) \\ C_3 & -C_4 & 0 \\ -W(t, e_w) & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_v \\ e_y \end{bmatrix} + \Psi_2\left(t, e_v, e_\theta\right) \quad (9)$$

where $W(t, e_w) = \text{diag}\{w_i(t, e_w)\}$, $C_3 := \text{diag}\{c_{3i}\}$, $C_4 := \text{diag}\{c_{4i}\}$ and the interconnection term is given by

$$\Psi_2 = \begin{bmatrix} (\text{Cos}e_\theta - I)v \\ \text{Sin}e_\theta v \\ 0_{n\times 1} \end{bmatrix} \quad (10)$$

where $v := [v_0 \cdots v_{n-1}]^\top$, $\text{Cos}e_\theta := \text{diag}\{\cos e_{i\theta}\}$ and $\text{Sin}e_\theta := \text{diag}\{\sin e_{i\theta}\}$. Note that each

$$v_i = e_{iv} + e_{(i-1)v} + \cdots + e_{1v} + v_0(t),$$

hence $v$ is a function of $t$ and $e_v$. Note also that $\Psi_2(t, e_v, 0) \equiv 0$. Furthermore, the heading error dynamics, given by equations (8c) and (8e) become

$$\begin{bmatrix} \dot{e}_\theta \\ \dot{e}_w \end{bmatrix} = \begin{bmatrix} 0 & -I \\ C_5 & -C_6 \end{bmatrix} \begin{bmatrix} e_\theta \\ e_w \end{bmatrix} \quad (11)$$

where $C_5 := \text{diag}\{c_{5i}\}$ and $C_6 := \text{diag}\{c_{6i}\}$.

**Proposition 1** *Consider the system* (1) *in closed loop with the controllers* (7) *with* $i \in \{1, ...n\}$ *where* $c_{3i}, c_{4i}, c_{5i}, c_{6i} > 0$ *and the references* $v_0$ *and* $w_0$ *satisfy*

$$\max\{\sup_{t\geq 0}|v_0(t)|\,,\ \sup_{t\geq 0}|w_0(t)|\,,\ \sup_{t\geq 0}|\dot{w}_0(t)|\} \leq b_\mu \quad (12)$$

*for some $b_\mu > 0$. Then, the origin of the closed-loop system is uniformly globally exponentially stable if and only if there exist positive constants $\mu$ and $T$ such that*

$$\mu \leq \int_t^{t+T} |w_0(\tau)|^2 \, d\tau \qquad \forall t \geq 0. \tag{13}$$

The condition (13) is known in the adaptive control literature as persistency of excitation. It is known that it is necessary and sufficient for exponential stability of a class of linear time-varying systems –see the Appendix.

*Proof:* The closed-loop dynamics is given by (9), (11) therefore, we must show that the origin of this system is uniformly globally exponentially stable and that persistency of excitation of $w_0$ is a necessary condition; we rely on Theorems 1 and 2 from the Appendix. Let us start by writing the closed-loop equations in a convenient form; define $x_2 := [e_\theta, e_w]^T$, and

$$f_2(t, x_2) := \begin{bmatrix} 0 & -I \\ C_5 & -C_6 \end{bmatrix} \begin{bmatrix} e_\theta \\ e_w \end{bmatrix}. \tag{14}$$

then, we see that (11) has the form (6b). Now, let

$$A := \begin{bmatrix} 0 & -I \\ C_3 & -C_4 \end{bmatrix}, \quad B(t, e_w) := \begin{bmatrix} W(t, e_w) \\ 0 \end{bmatrix}$$

and let

$$f_1(t, x_1) := \begin{bmatrix} A & B(t,0) \\ -B(t,0)^\top & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} e_x \\ e_v \end{bmatrix} \\ e_y \end{bmatrix} \tag{15}$$

where $x_1 := [e_x, e_v, e_y]^\top$. Notice that

$$B(t, 0) := \begin{bmatrix} W(t,0) \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} w_0(t)$$

and $B(t,0)^\top B(t,0) = w_0(t)^2 I$.

Furthermore, let us introduce

$$g(t,x) = \begin{bmatrix} 0 & B(t,e_w) - B(t,0) \\ -B(t,e_w) + B(t,0) & 0 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} e_x \\ e_v \end{bmatrix} \\ e_y \end{bmatrix} + \Psi_2(t, e_v, e_\theta)$$

Notice that $x_2 = 0$ implies that $e_w = 0$, $e_\theta = 0$ hence,

$$g(t,x)\Big|_{x_2=0} = \Psi_2(t, e_v, 0) = 0.$$

We are ready to invoke Theorem 1. Assumption A1 holds with the quadratic function

$$V(t, x_1) = \frac{1}{2} \left[ e_x^\top C_3 e_x + e_y^\top C_3 e_y + |e_v|^2 \right] \tag{16}$$

so the conditions (19) and (20) hold with $c_2 = \max\{c_{3i}, c_{4i}\}/2$, $\eta = 1$ and $c_1 = 2c_2 / \min\{c_{3i}, c_{4i}\}$. Furthermore, the total time-derivative of $V$ along the trajectories of $\dot{x}_1 = f_1(t, x_1)$ with the latter defined in (15) yields

$$\dot{V}_{(30)}(t, x_1) = -e_v^T C_4 e_v \leq 0$$

To see that Assumption A2 holds observe that $x_2 = 0$ implies that $g = 0$ for any $t \geq 0$ and $x_1 \in \mathbb{R}^{3n}$ and $\Psi_2$ is linear in $[e_x \, e_v \, e_y]$ and uniformly bounded in $t$ –see (10). ■

## IV. SIMULATION RESULTS

To illustrate our theoretical findings we present some simulation results, obtained using SIMULINK™of MATLAB™. We consider a team of 3 mobile robots where one of them is the global leader which knows the reference trajectory and the other two as followers.

In the first stage of the simulation, the desired formation shape of the mobile robots is in triangular form with following initial condition; $[x_1(0), y_1(0), \theta_1(0)]^T = [0, -4, 3\pi/8]$, $[x_2(0), y_2(0), \theta_2(0)]^T = [-3.5, -7, \pi/2]$ and $[x_3(0), y_3(0), \theta_3(0)]^T = [-5, -1, \pi/3]$ and the triangular formation shape is obtained via $[d_{x1,2}, d_{y1,2}] = [\sqrt{3}, 1]$ and $[d_{x2,3}, d_{y2,3}] = [0, -2]$. In order to show the flexibility of the formation, after an arbitrary period of time, we allow the formation shape to change from triangular to line with a following desired distance between robots, $[d_{x1,2}, d_{y1,2}] = [0, 2]$ and $[d_{x2,3}, d_{y2,3}] = [0, 2]$. In order to obtain the reference trajectory of the leader robot, we set the linear and angular velocities as $[v_0(t), w_0(t)] = [15 \ m/s, \ 3 \ rad/s]$.

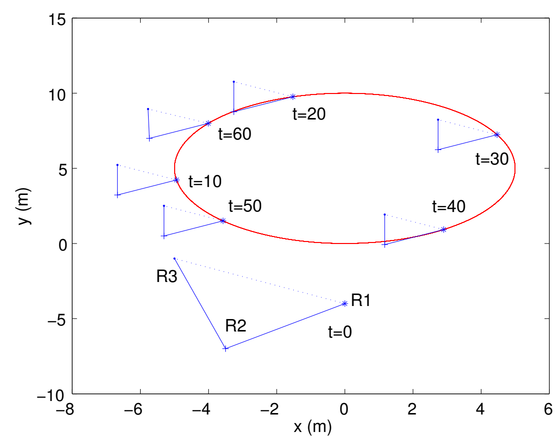


Fig. 3. Motion and relative positioning of the robots in triangular formation on the plane.

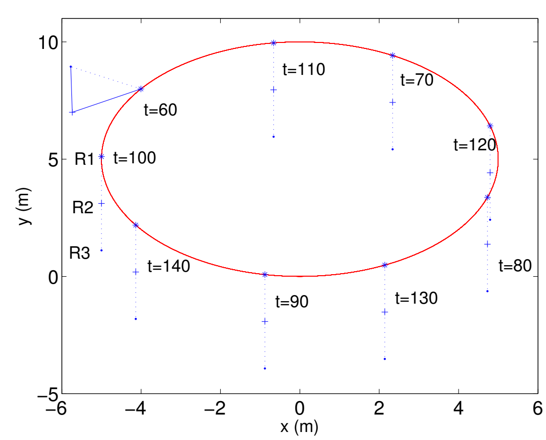


Fig. 4. Motion and relative positioning of the robots in alined formation on the plane.

The control parameters are set to $C_3 = diag\{12, 17, 17\}, C_4 = diag\{5\}, C_5 = C_6 = diag\{10\}$. Briefly, we present that the robots reach to the desired triangular formation and change the shape to the line-form after 60[s] with a satisfactory performance. The simulation results are showed in Figures 5–7.

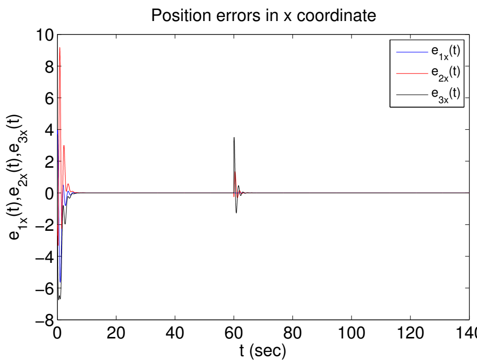


Fig. 5. Position errors in $x$ coordinates with dynamic control algorithm.

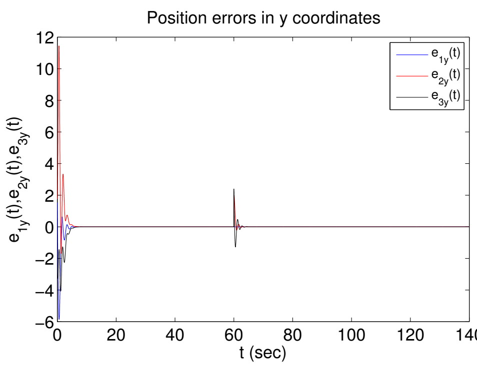


Fig. 6. Position errors in $y$ coordinates with dynamic control algorithm.

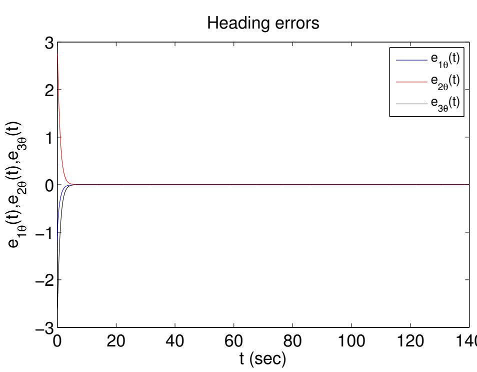


Fig. 7. Heading errors with dynamic control algorithm.

## V. Conclusion

We presented a simple consensus controller based on a spanning-tree topology. That is, each robot except the root and the leaf, is master to a follower and slave to a leader. Consensus is ensured based on a condition of persistency of excitation, imposed on the reference angular velocity profile that is, the angular velocity of the virtual leader robot. This rules out straight-line paths. The extension to this case involves the use of a condition of persistency of excitation which depends also on the states and is under investigation.

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## VI. Appendix

We present below two theorems on stability, paraphrased from the literature for the purposes of this note; the first is on cascaded systems, the second establishes stability for a class of adaptive control systems.

Consider the system (6) where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $x \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$. The function $f_1$ is locally Lipschitz in $x_1$ uniformly in $t$ and $f(\cdot, x_1)$ is continuous, $f_2$ is continuous and locally Lipschitz in $x_2$ uniformly in $t$, $g$ is continuous in $t$ and once differentiable in $x$. The theorem given below which is reminiscent of the results originally presented in [20] establishes unifom global exponential stability of the cascaded non-autonomous systems.

**Theorem 1** *Let the respective origins of*

$$\Sigma_1 : \ \dot{x}_1 = f_1(t, x_1) \tag{17}$$

$$\Sigma_2 : \ \dot{x}_2 = f_2(t, x_2) \tag{18}$$

*be uniformly globally exponentially stable and let the following assumptions hold.*

(A1) *There exist a Lyapunov function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for (24) which is positive definite, radially unbounded,*

$$\dot{V}_{(24)}(t, x_1) := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} f_1(t, x_1) \leq 0$$

*and constants $c_1, c_2, \eta > 0$ such that*

$$\left| \frac{\partial V}{\partial x_1} \right| |x_1| \leq c_1 V(t, x_1) \qquad \forall \ |x_1| \geq \eta \tag{19}$$

$$\left| \frac{\partial V}{\partial x_1} \right| \leq c_2 \qquad \forall \ |x_1| \leq \eta \tag{20}$$

(A2) *There exist two continuous functions $\theta_1, \theta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $g(t, x_1, x_2)$ satisfies*

$$|g(t, x_1, x_2)| \leq \theta_1(|x_2|) + \theta_2(|x_2|) \ \ |x_1| \tag{21}$$

*Then, the origin of the cascaded system (22), (23) is uniformly globally exponentially stable.*

Note that Assumption A1 holds for quadratic functions; let $V(t, x_1) := x_1^\top P x_1$ with $P$ positive definite then

$$\left| \frac{\partial V}{\partial x_1} \right| = |P x_1| \leq \lambda_M(P) |x_1| \leq \lambda_M(P) \qquad \forall \ |x_1| \leq 1$$

while

$$\left| \frac{\partial V}{\partial x_1} \right| |x_1| \leq \lambda_M(P) |x_1|^2 \leq \frac{\lambda_M(P)}{\lambda_m(P)} V(t, x_1) \quad \forall \, x_1 \in \mathbb{R}^{n_1}.$$

Roughly speaking, Assumption A2 holds if $g(t, x_1, x_2)$ has linear growth order with respect to $x_1$, uniformly in $t$ for each fixed $x_2$.

The following theorem is restated from the literature on adaptive control, see for instance [21].

**Theorem 2** *For the system*

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & B(t) \\ C(t)^\top & 0 \end{bmatrix} \tag{22}$$

*Let $A$ be Hurwitz, let $P = P^\top > 0$ be such that $A^\top P + PA = -Q$ is negative definite and $PB = C^\top$. Assume that $B$ is uniformly bounded and has a continuous uniformly bounded derivative. Then, the origin is uniformly globally exponentially stable if and only if $B$ is persistently exciting that is, if there exist positive constants $\mu$ and $T$ such that*

$$\mu_1 I \leq \int_t^{t+T} B(\tau)^\top B(\tau) d\tau \qquad \forall t \geq 0. \tag{23}$$